

Moduli space of genus one curves on certain del Pezzo threefolds

BGS

§ The Problem Let X be a smooth Fano variety / $\mathbb{Q}$

Q: Can we describe the moduli space of curves on X ?

- Morphism space $\text{Mor}(C, X)$, C sm genus g curve

$$\text{Mor}(C, X) = \bigsqcup_{\alpha \in A_1(X)} \text{Mor}(C, X, \alpha)$$

$$\text{Mor}(C, X, \alpha)(\mathbb{Q}) = \{ f: C \rightarrow X \mid f_*C = \alpha \}$$

- Kontsevich space of stable maps

$$\overline{\mathcal{M}}_{g,0}(X) = \bigsqcup_{\alpha \in A_1(X)} \overline{\mathcal{M}}_{g,0}(X, \alpha) \quad \begin{array}{l} \text{DM stack} \\ \text{proj CMS} \end{array}$$

$$\overline{\mathcal{M}}_{g,0}(X, \alpha)(\mathbb{Q}) = \left\{ f: C \rightarrow X \mid \begin{array}{l} f_*C = \alpha \\ f \text{ is a stable map} \end{array} \right\}$$

a stable map: C is a conn. nodal genus g

if $C_i \subset C$ s.t. C_i is contracted.
irred

then if $g(C_i) = 0$, then C_i contains
at least 3 nodes

if $g(C_i) = 1$, then C_i contains
at least 1 node

Ask: - What are the number of irreducible comps?
- What are the dimension of these comps?

- Useful for :
- enumerativity of Gromov-Witten inv
 - compute the asymptotic formula of the number of rational pts on X of bounded height

Some Known Results # of irreducible ceps & dimension

$$\overline{M}_{0,0}(X) \quad \overline{M}_{0,0}(\mathbb{P}^1, X)$$

- general smooth hypersurfaces $\deg d \subset \mathbb{P}^n$

Harris-Roth-Starr 04 when $d < \frac{n+1}{2}$

Riedl-Yang 19 when $d \leq n-2$

- all cubic hypersurfaces $\dim \geq 3$

Starr, Coskun-Starr 09

- del Pezzo surfaces

Testa 05, Beheshti-Lehmann-Riedl-Torricelli 23

- smooth Fano threefolds

BRLT 22, Burke-Torricelli 23

"Geometric Methods"

Difficulties to extend to $g > 0$

Ex. $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ plane cubics

- $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ stable maps w/ irred domain
smooth cubics $\dim 9$

$$\left\{ \begin{array}{c} \mathbb{P}^E \\ \downarrow f \\ \mathbb{P}' \end{array} \xrightarrow{f} \begin{array}{c} \text{loop} \\ \uparrow f(E) \\ \downarrow f(\mathbb{P}') \end{array} \right\} \text{ locus has dim } 10$$

"
 W

$$W \cap \overline{\mathcal{M}_{1,0}(\mathbb{P}^2, 3)} \neq \emptyset$$

Result $g=1$. X sm del Pezzo threefold w/ $\text{Pic } X = \mathbb{Z}H$.
 $H^3 > 3$. $-K_X = 2H$.

Thm (Iskovskikh)

Let X be as above. Then $H^3 \in \{3, 4, 5\}$.

$H^3 = 5$: X is a section of $\text{Gr}(1, 4) \subset \mathbb{P}^9$ by a general codim 3 linear section

$H^3 = 4$: X is a sm complete intersection of 2 quadrics in $\mathbb{P}^5$

$H^3 = 3$: X is a sm cubic threefold

$\widetilde{\mathcal{M}}_{1,0}(X, e)$: closure of $\mathcal{M}_{1,0}(X, e)$ in $\overline{\mathcal{M}}_{1,0}(X, e)$

Thm^A (F-Okumura)

Let X be as above. Then for H -deg $e \geq 5$, the space $\widetilde{\mathcal{M}}_{1,0}(X, e)$ has exactly 2 irred comps.

R_e : gen parametrises maps that are birational onto their images dim $2e$

N_e : covers of lines dim $2e+2$

Rank, known for $e \leq 4$. But there is an extra component Q_4 : double cover of conics

Thm^B (F-Okumura)

Let H -deg $e \geq 6$ and E to be a general elliptic curve. Then $\text{Mor}(E, X, e)$ consists of exactly 2 irred comps.

R_e, N_e : some descriptions
dim $2e, 2e+2$

§. Geometric Mordell Conjecture (Lehmann-Tanimoto ...)

C sm genus g curve / $\mathbb{F}_q$ $K = K(C)$

X sm Fano var / $\mathbb{F}_q$

the number of $X_K(K) \longleftrightarrow \text{Mor}(C, X)(\mathbb{F}_q)$

$x \in X_K(K) \longleftrightarrow \tilde{x}: C \rightarrow X$
val criterion

define $H(x) = q^{-K_X \cdot \tilde{x}^* C}$

Mordell's Conj: There exists some exceptional set $Z \subset X_K(K)$

$\# \{x \in X_K(K) \setminus Z \mid H(x) \leq B\} \sim c \cdot B^{\frac{p(X)-1}{2}} \cdot (\log B)$
 $B \rightarrow \infty$

Badzyrev 88: this asymptote is derived

when $\text{Mor}(C, X, \alpha)$ is irred of the
exp dim make sense / ϕ

Geometric Manin Conj / 4

comps $\text{Mor}(C, X) =$ Exceptional Components $\cup$ Manin Comps
 can have larger $\swarrow$ gen smooth
 dimension & exp dimension
 can be non-reduced
 $\sim \mathbb{Z}$ exceptional set

Def. Let $M \subset \text{Mor}(C, X)$ an irred comp.
 then M is exceptional if there exists
 a generically finite morphism $f: Y \rightarrow X$ st.

big & nef

$$\alpha(X, L)$$

$$= \inf \{ t \mid K_X + tL \in \overline{\text{Eff}}^1(X) \}$$

1. f is not both birational & dominant

$$2. \alpha(Y, -f^*K_X) \geq \alpha(X, -K_X) = 1$$

3. $\exists$ some $N \subset \text{Mor}(C, Y)$ st.

$$f_*: N \rightarrow M \text{ is dominant}$$

motivated by Manin's Conj.

$$c \cdot B \cdot \frac{1 - \alpha(X, -K_X)}{(\log B)^{\alpha(X) - 1}}$$

Otherwise, M is Manin.

(Exceptional Set) Exceptional comps are controlled by
 the α -invariant

(Uniqueness) for each nef $\alpha \in A_1(X)$, sufficiently positive
 there is a unique Manin component parametrizing
 maps with class α

free curves

$$f: C \rightarrow X,$$

$$f^*T_X \text{ gen glob gen} \\ h^1(f^*T_X) = 0$$

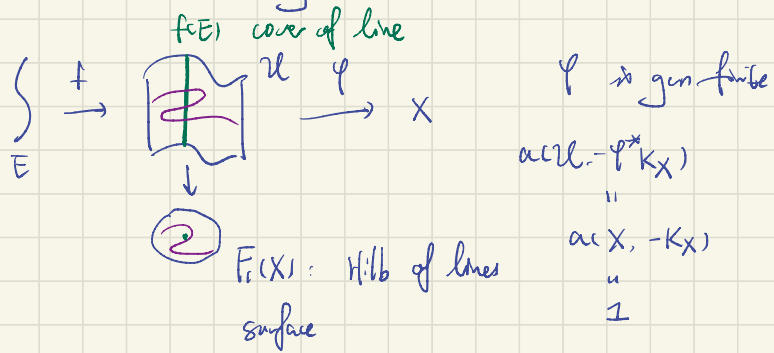
gen param free curves

(Stability) Main comps exhibit homological / motivic stability

§ Proof of the Theorem

Step I. Classify the excep comps

Thm. If $M \subset \text{Mor}(E, X)$ is excep. then the maps param by M factor through the universal family of lines $\mathcal{U}$ on X .



$\overline{F}_1(X)$ is irreducible

$\Rightarrow$ deg e covers of lines form an ined comp

explicit geom of $\mathcal{U}$ and $F_1(X)$, dim count

$\Rightarrow$ non-covers of lines do not form comps

$\Rightarrow \text{Mor}^{\text{line}}(E, X, e)$ is irreducible

as E varies in moduli

$\Rightarrow \tilde{M}_{1,0}(X, e)_{\text{lines}}$ is irred.

Step II Classify the Mori comp.

Thm C Let $H\text{-deg } e \geq 6$. A Mori component
generally parametrizes free curves and
they are open subsets of the fibres
of the map:

stable maps
w/ 1 marked points
general member in
an embedding

$$\leftarrow \tilde{M}_{1,1}^{\text{bir}}(X, e) \xrightarrow{\pi} \overline{M}_{1,1}$$

$\Rightarrow$ classify $\tilde{M}_{1,0}^{\text{bir}}(X, e)$

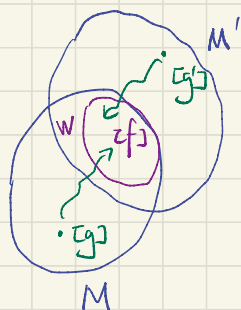
Thm (Movable Bend-and-Break)

Let $M \subset \tilde{M}_{1,0}^{\text{bir}}(X, e)$. The M parametrizes a

stable map $f: E \cup \mathbb{P}^1 \rightarrow X$

s.t. $f|_E$ and $f|_{\mathbb{P}^1}$ are free curves
and horizontal onto their images

i.e. $[f]$ is a smooth pt of $\tilde{M}_{1,0}(X, e)$



Induction on the irred
of lower degree comps

$$\sim^{br} \overline{M}_{1,0}(X, e'), \quad e' < e$$

together w/ Stein, LT

$$\overline{M}_{0,0}^{br}(X, e-e') \text{ is irred}$$

Since $[f]$ is a sm pt
of $\overline{M}_{1,0}(X, e)$

$$W = \sim^{br} \overline{M}_{1,1}(X, e') \times \overline{M}_{0,1}^{br}(X, e-e')$$

is irred.

$$\Rightarrow M = M'$$

$$\Rightarrow \sim^{br} \overline{M}_{1,0}(X, e) \text{ is irred.}$$

$$W = \left\{ \begin{array}{c} \text{curve } E \text{ meeting } \mathbb{P}^1 \text{ at } p \\ \rightarrow X \end{array} \right\}$$

$\leadsto$ go back to $\text{Mor}^{br}(E, X)$

Consider the Stein factorization

$$\begin{array}{ccc} \sim^{br} \overline{M}_{1,1}(X, e) & & \\ \pi \downarrow & \searrow D & \\ \mathbb{P}^1 \cong \overline{M}_{1,1} & \swarrow h & \end{array}$$

if h has $\deg = 1$

then since # of irred comps

is upper semi-cts.

then for general $(E, p) \in \overline{M}_{1,1}$

$$\pi^{-1}(E, p) \text{ is irred.}$$

$$\xRightarrow{\text{Thm C}} \text{Mor}^{br}(E, X) \text{ is irred.}$$

use deformation argument to show
that $\deg h = 1$.